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Quantitative Methods

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QM-A3-engb 1/2016 (1022)

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Quantitative Methods

The Quantitative Methods programme is written by David Targett, Professor of Information Systems at the School of Management, University of Bath and formerly Senior Lecturer in Decision Sciences at the London Business School. Professor Targett has many years' experience teaching executives to add numeracy to their list of management skills and become balanced decision makers. His style is based on demystifying complex techniques and demonstrating clearly their practical relevance as well as their shortcomings. His books, including *Coping with Numbers* and *The Economist Pocket Guide to Business Numeracy*, have stressed communication rather than technical rigour and have sold throughout the world.

He has written over fifty case studies which confirm the increasing integration of Quantitative Methods with other management topics. The cases cover a variety of industries, illustrating the changing nature of Quantitative Methods and the growing impact it is having on decision makers in the Information Technology age. They also demonstrate Professor Targett's wide practical experience in international organisations in both public and private sectors.

One of his many articles, a study on the provision of management information, won the Pergamon Prize in 1986.

He was part of the team that designed London Business School's highly successful part-time MBA Programme of which he was the Director from 1985 to 1988. During this time he extended the international focus of the teaching by leading pioneering study groups to Hong Kong, Singapore and the United States of America. He has taught on all major programmes at the London Business School and has developed and run management education courses involving scores of major companies including:

British Rail
Citicorp
Marks and Spencer
Shell

First Published in Great Britain in 1990.

Contents

PART 1

Introduction and Background

Module 1

Introducing Statistics: Some Simple Uses and Misuses

Contents

Prerequisite reading: None

Learning Objectives

This module gives an overview of statistics, introducing basic ideas and concepts at a general level, before dealing with them in greater detail in later modules. The purpose is to provide a gentle way into the subject for those without a statistical background, in response to the cynical view that it is not possible for anyone to read a statistical text unless they have read it before. For those with a statistical background, the module will provide a broad framework for studying the subject.

1.1 Introduction

The word **statistics** can refer to a collection of numbers or it can refer to the science of studying collections of numbers. Under either definition the subject has received far more than its share of abuse ('lies, damned lies…'). A large part of the reason for this may well be the failure of people to understand that statistics is like a language. Just as verbal languages can be misused (for example, by politicians and journalists?) so the numerical language of statistics can be misused (by politicians and journalists?). To blame statistics for this is as sensible as blaming the English language when election promises are not kept.

One does not have to be skilled in statistics to misuse them deliberately ('figures can lie and liars can figure'), but misuses often remain undetected because fewer people seem to have the knowledge and confidence to handle numbers than have similar abilities with words. Fewer people are numerate than are literate. What is needed to see through the misuse of statistics, however, is common sense with the addition of only a small amount of technical knowledge.

The difficulties are compounded by the unrealistic attitudes of those who do have statistical knowledge. For instance, when a company's annual accounts report that the physical stock level is £34 236 417 (or even £34 236 000), it conveys an aura of truth because the figure is so precise. Accompanying the accountants who estimated the figure, one may have thought that the method by which the data were collected did not warrant such precision. For market research to say that 9 out of 10 dogs prefer Bonzo dog food is also misleading, but in a far more overt fashion. The statement is utterly meaningless, as is seen by asking the questions: 'Prefer it to what?', 'Prefer it under what circumstances?', '9 out of which 10 dogs?'

Such examples and many, many others of greater or lesser subtlety have generated a poor reputation for statistics which is frequently used as an excuse for remaining in ignorance of it. Unfortunately, it is impossible to avoid statistics in business. Decisions are based on information; information is often in numerical form. To make good decisions it is necessary to organise and understand numbers. This is what statistics is about and this is why it is important to have some knowledge of the subject.

Statistics can be split into two parts. The first part can be called **descriptive statistics**. Broadly, this element handles the problem of sorting a large amount of collected data in ways which enable its main features to be seen immediately. It is concerned with turning numbers into real and useful information. Included here are simple ideas such as organising and arranging data so that their patterns can be seen, summarising data so that they can be handled more easily and communicating data to others. Also included is the now very important area of handling computerised business statistics as provided by management information systems and decision support systems.

The second part can be referred to broadly as **inferential statistics**. This element tackles the problem of how the small amount of data that has been collected (called the **sample**) may be analysed to infer general conclusions about the total amount of similar data that exist uncollected in the world (called the **population**). For instance, opinion polls use inferential statistics to make statements about the opinions of the whole electorate of a country, given the results of perhaps just a few hundred interviews.

Both types of statistics are open to misuse. However, with a little knowledge and a great deal of common sense, the errors can be spotted and the correct procedures seen. In this module the basic concepts of statistics will be introduced. Later, some abuses of statistics and how to counter them will be discussed.

The first basic concept to look at is that of probability, which is fundamental to statistical work. Statistics deals with approximations and 'best guesses' because of the inaccuracy and incompleteness of most of the data used. It is rare to make statements and draw conclusions with certainty. Probability is a way of quantifying the strength of belief in the information derived and the conclusions drawn.

1.2 Probability

All future events are uncertain to some degree. That the present government will still be in power in the UK in a year's time (given that it is not an election year) is likely, but far from certain; that a communist government will be in power in a year's time is highly unlikely, but not impossible. Probability theory enables the difference in the uncertainty of events to be made more precise by measuring their likelihood on a scale.

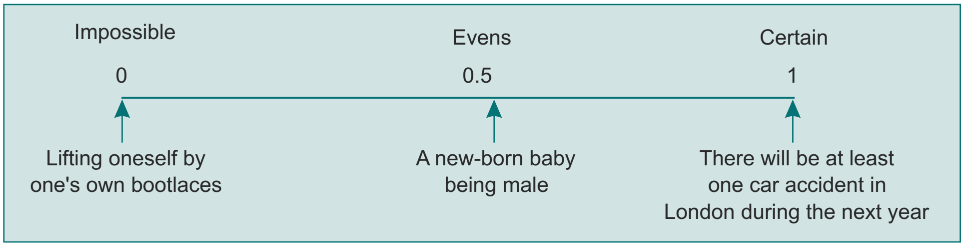

Figure 1.1 **Probability scale**

The scale is shown in Figure 1.1. At one extreme, impossible events (e.g. that you could swim the Atlantic) have probability zero. At the other extreme, completely certain events (e.g. that you will one day die) have probability one. In between are placed all the neither certain nor impossible events according to their likelihood. For instance, the probability of obtaining a head on one spin of an unbiased coin is 0.5; the probability of one particular ticket winning a raffle in which there are 100 tickets is 0.01.

As a shorthand notation 'the probability of an event A is 0.6' is written in this way:

$$P(\mathrm{A}) = 0.6$$

1.2.1 Measurement of Probability

There are three methods of calculating a probability. The methods are not alternatives since for certain events only one particular method of measurement may be possible. However, they do provide different conceptual ways of viewing probability. This should become clear as the methods are described.

(a) **A priori approach.** In this method the probability of an event is calculated by a process of logic. No experiment or judgement is required. Probabilities involving coins, dice and playing cards can fall into this category. For example, the probability of a coin landing 'heads' can be calculated by noting that the coin has two sides, both of which are equally likely to fall upwards (pedants, please note: assume it will not come to rest on its rim). Since the coin must fall with one side upwards, the two events must share equally the total probability of 1.0. Therefore:

$$P(\text{Heads}) = 0.5$$
$$P(\text{Tails}) = 0.5$$

(b) **'Relative frequency' approach.** When the event has been or can be repeated a large number of times, its probability can be measured from the formula:

$$P(\text{Event}) = \frac{\text{No. times event occurs}}{\text{No. trials}}$$

For example, to estimate the probability of rain on a given day in September in London, look at the last 10 years' records to find that it rained on 57 days. Then:

$$\begin{aligned} P(\text{Rain}) &= \frac{\text{No. days rain recorded}}{\text{Total no. days } (=10\times30)} \\ &= \frac{57}{300} \\ &= 0.19 \end{aligned}$$

(c) **Subjective approach.** A certain group of statisticians (Bayesians) would argue that the degree of belief that an individual has about a particular event may be expressed as a probability. Bayesian statisticians argue that in certain circumstances a person's subjective assessment of a probability can and should be used. The traditional view, held by classical statisticians, is that only objective probability assessments are permissible. Specific areas and techniques that use subjective probabilities will be described later. At this stage it is important to know that probabilities can be assessed subjectively but that there is discussion amongst statisticians as to the validity of doing so. As an example of the subjective approach, let the event be the achievement of political unity in Europe by the year 2020 AD. There is no way that either of the first two approaches could be employed to calculate this probability. However, an individual can express his own feelings on the likelihood of this event by comparing it with an event of known probability: for example, is it more or less likely than obtaining a head on the spin of a coin? After a long process of comparison and checking, the result might be:

$$P(\text{Political unity in Europe by 2020 AD}) = 0.10$$

The process of accurately assessing a subjective probability is a field of study in its own right and should not be regarded as pure guesswork.

The three methods of determining probabilities have been presented here as an introduction and the approach has not been rigorous. Once probabilities have been calculated by whatever method, they are treated in exactly the same way.

Examples

1. What is the probability of throwing a six with one throw of a die?
   With the a priori approach there are six possible outcomes: 1, 2, 3, 4, 5 or 6 showing. All outcomes are equally likely. Therefore:
   $$P(\text{throwing a 6}) = \frac{1}{6}$$
2. What is the probability of a second English Channel tunnel for road vehicles being completed by 2025 AD?
   The subjective approach is the only one possible, since logical thought alone cannot lead to an answer and there are no past observations. My assessment is a small one, around 0.02.
3. How would you calculate the probability of obtaining a head on one spin of a biased coin?

The a priori approach may be possible if one had information on the aerodynamical behaviour of the coin. A more realistic method would be to conduct several trial spins and count the number of times a head appeared:

$$P(\text{obtaining a head}) = \frac{\text{No. observed heads}}{\text{No. trial spins}}$$

4. What is the probability of drawing an ace in one cut of a pack of playing cards?
   Use the a priori method. There are 52 possible outcomes (one for each card in the deck) and the probability of picking any one card, say the ace of diamonds, must therefore be 1/52. There are four aces in the deck, hence:

$$P(\text{drawing an ace}) = \frac{4}{52} = \frac{1}{13}$$

1.3 Discrete Statistical Distributions

Probability makes it possible to study another essential element of statistical work: the statistical distribution. It can be thought of either as one of the first steps in descriptive statistics or, alternatively, as a cornerstone of inferential statistics. It will first be developed as a descriptive technique. Suppose there is a collection of data, which initially might appear as in Figure 1.2.

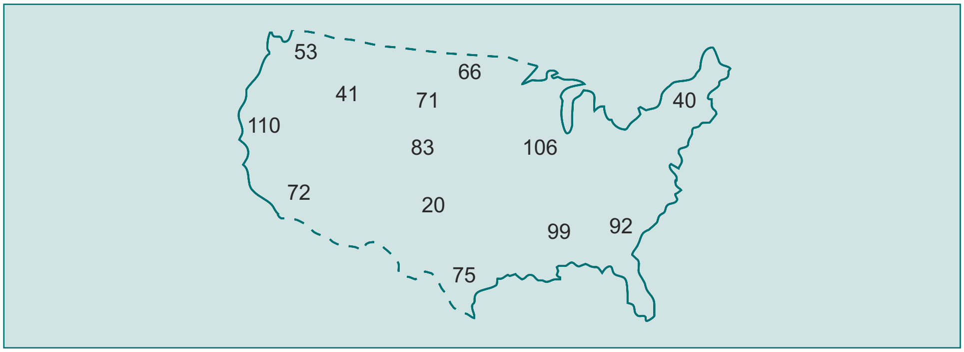

Figure 1.2 USA sales data

The numbers are all measurements of a variable. A variable is just what the word implies. It is some entity which can be measured and for which the measurement varies when several observations are made. The variable might be the number of serious crimes in each French département or the heights of all 20-year-old males in Sweden. Figure 1.2 shows the annual sales (in thousands) of a brand of tinned sweetcorn in different sales regions of the USA. The numbers are referred to as **observations** or **data points**.

It is little more than a mess. A mess can take on different forms, of course. The first sight of a particular set of data may be a pile of dusty production dockets or it may be a file of handwritten invoices, but it is always likely to be some sort of mess. A first attempt to sort it out might be to arrange the numbers in order as in Table 1.1.

Table 1.1 **Columns of numbers**

.	52	59	66
.	54	60	66
41	55	60	.
43	56	60	.
45	57	61	.
46	57	62	.
48	58	62	
49	58	63	
49	58	65	
50	59	65	

Table 1.1 is an ordered array. The numbers look neater now but it is still not possible to get a feel for the data (the average, for example) as they stand. The next step is to classify the data and then arrange the classes in order. Classifying means grouping the numbers in bands (e.g. 50–54) to make them easier to handle. Each class has a frequency, which is the number of data points that fall within that class. This is called a **frequency table** and is shown in Table 1.2. This shows that seven data points were greater than or equal to 40 but less than 50, 12 were greater than or equal to 50 but less than 60 and so on. There were 100 data points in all.

Table 1.2 **A frequency table**

Class	Frequency
$40 \leq x < 50$	7
$50 \leq x < 60$	12
$60 \leq x < 70$	22
$70 \leq x < 80$	27
$80 \leq x < 90$	19
$90 \leq x < 100$	10
$100 \leq x < 110$	3
Total frequency	100

Note: ≤ means 'less than or equal to'; < means 'less than'.

It is now much easier to get an overall conception of what the data mean. For example, most of the numbers are between 60 and 90 with extremes of 40 and 110. Of course, it is likely that at some time there may be a need to perform detailed calculations with the numbers to provide specific information, but at present the objective is merely to get a feel for the data in the shortest possible time. Another arrangement with greater visual impact, called a **frequency histogram**, will help meet this objective.

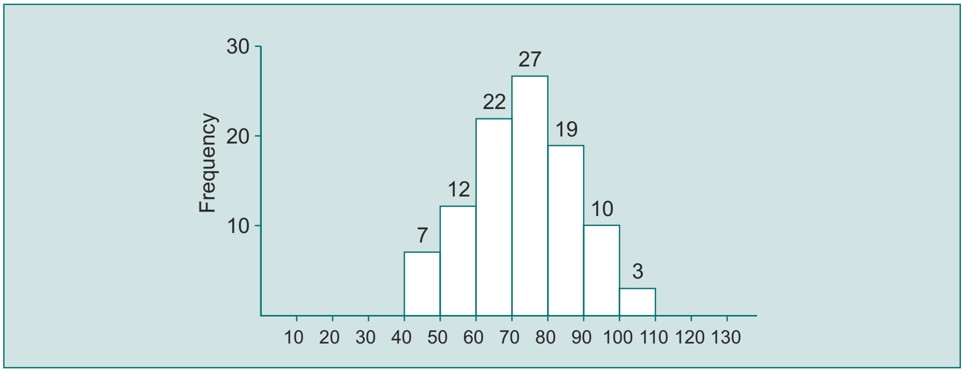

Figure 1.3 **A frequency histogram**

The transition from Table 1.2 to Figure 1.3 is simple and obvious, yet with the frequency histogram one can see immediately what the data are like. The numbers are spread symmetrically over a range from 40 to just over 100 with the majority falling around the centre of the range.

As a descriptive device the frequency histogram works well and it is not necessary to refine it further. If, on the other hand, there are analytical objectives, the histogram of Figure 1.3 would be developed into a statistical distribution. To be strictly accurate, all configurations dealt with are statistical distributions, but it is the most manageable and generally accepted version that is sought.

To carry out this development, notice first the connection between frequencies and probabilities via the 'relative frequency' approach to probability calculations. The probability that any randomly selected measurement lies within a particular class interval can be calculated as follows:

$$P(\text{number lies within class } x) = \frac{\text{Frequency class } x}{\text{Total frequency}} = \frac{\text{Column height } x}{\text{Total frequency}}$$

e.g.

$$P(40 \leq x < 50) = \frac{7}{100} = 0.07$$

The frequency histogram can then be turned into a **probability histogram** by writing the units of the vertical axis as probabilities (as calculated above) instead of frequencies. The shape of the histogram would remain unaltered. Once the histogram is in the probability form it is usually referred to as a **distribution**, in this case a **discrete distribution**. A variable is discrete if it is limited in the values it can take. For example, when the data are restricted to classes (as above) the variable is discrete. Also, when a variable is restricted to whole numbers only (an integer variable), it is discrete.

The probability histogram makes it easier to work out the probabilities associated with amalgams of classes. For instance, if the probabilities of two of the classes are:

$$P(50 \leq x < 60) = 0.12$$

$$P(60 \leq x < 70) = 0.22$$

then:

$$\begin{aligned} P(50 \leq x < 70) &= 0.12 + 0.22 \\ &= 0.34 \end{aligned}$$

This is true whether working in probabilities or the frequencies from which they were derived.

Examples

From the data in Figure 1.3, what are the following probabilities?

1. $P(80 \leq x < 100)$
2. $P(x < 70)$
3. $P(60 \leq x < 100)$

Answers

1.

$$\begin{aligned} P(80 \leq x < 100) &= P(80 \leq x < 90) + P(90 \leq x < 100) \\ &= \frac{19}{100} + \frac{10}{100} \\ &= 0.19 + 0.10 \\ &= 0.29 \end{aligned}$$

2.

$$\begin{aligned} P(x < 70) &= P(x \leq 50) + P(50 \leq x < 60) + P(60 \leq x < 70) \\ &= 0.07 + 0.12 + 0.22 \\ &= 0.41 \end{aligned}$$

3.

$$\begin{aligned} P(60 \leq x < 100) &= 0.22 + 0.27 + 0.19 + 0.10 \\ &= 0.78 \end{aligned}$$

1.4 Continuous Statistical Distributions

To summarise progress so far – there is a probability histogram of a variable, from which can be determined the probability that any one measurement of the variable will fall within one of the classes of the histogram. Such a distribution is a discrete distribution. It is a distribution because the variable is distributed across a range of values; it is discrete because the values the variables take are in steps rather than smoothly following one another.

A **continuous variable** is not limited in the values it can take. It can be whole numbers, and all values in between; it does not group data in classes, but distinguishes between numbers such as 41.73241 and 41.73242. The distribution formed by a continuous variable is a **continuous distribution**. It can be thought of as an extension of a discrete distribution. The extension process is as follows. (The process is to illustrate the link between discrete and continuous distributions: it is not a procedure that would ever need to be carried out in practice.)

A discrete distribution like Figure 1.3 is reproduced in Figure 1.4(a). The column widths are progressively reduced. In (b) the column widths have been halved; for example, the class $50 \leq x < 60$ is divided into two classes, $50 \leq x < 55$ and $55 \leq x < 60$. In (c) the classes have been further subdivided. As the process continues, the distribution becomes smoother, until, ultimately, the continuous distribution (d) will be achieved.

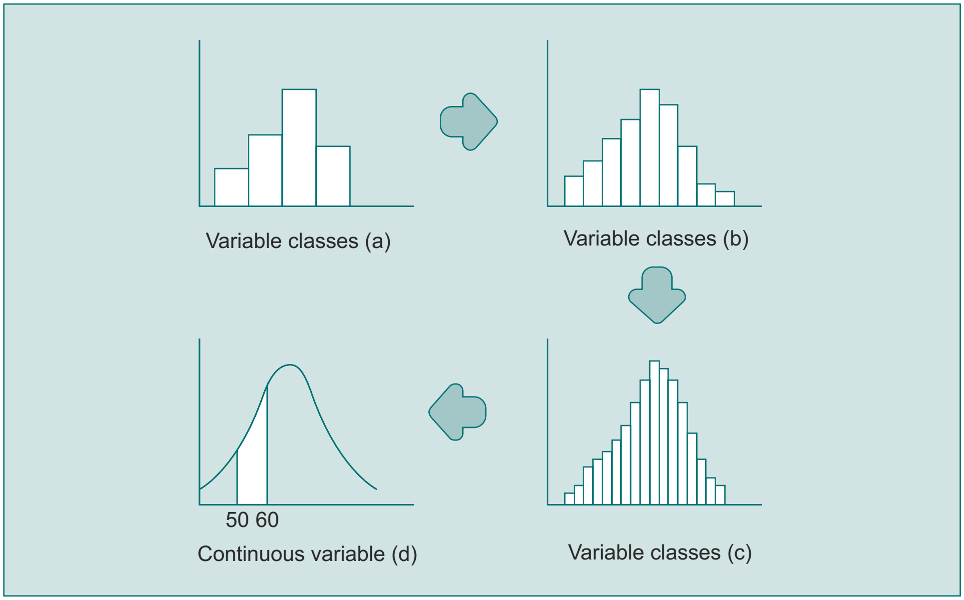

Figure 1.4 Discrete to continuous

There is now a difficulty concerning the measurement of probabilities. In the discrete distribution Figure 1.4(a), the probabilities associated with different values of the variable were equal to the column height. If column heights continue to be equal to probabilities, the process (a) → (b) → (c) → (d) would result in flatter and flatter distributions. Figure 1.4(d) would be completely flat since the probability associated with the now distinct values such as 41.73241 and 41.73242 must be infinitesimally small. The problem is overcome by measuring probabilities in a continuous distribution by *areas*. For example, $P(50 \leq x < 60)$ is the area under the part of the curve between 50 and 60, and shaded in Figure 1.4(d).

The argument for using areas is this. In Figure 1.4(a) the column widths are all the same; therefore probability could be measured just as well by area as by height. Figure 1.5 gives an example of what happens in the move from (a) to (b), when the classes are halved. It is supposed that the original data are such that the probabilities for the new classes can be calculated from them.

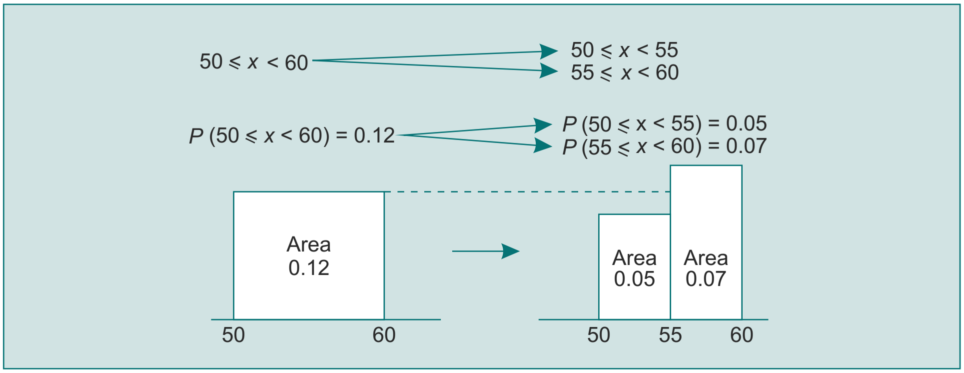

Figure 1.5 **Reducing class sizes**

Using areas to measure probabilities, the column heights of the new classes are approximately the same as those of the original. The lower probabilities for the new classes are reflected in the halving of the column widths, rather than changes in the heights. As the subdivision process continues, there is no tendency for the distribution to become flatter. In this way a continuous distribution can have a definite shape which can be interpreted in the same way as the shape of a discrete distribution, but its probabilities are measured from areas. Just as the column heights of a discrete distribution sum to 1 (because each observation *certainly* has some value), so the total area of a continuous distribution is 1.

The differences between discrete and continuous distributions are summarised in Table 1.3.

Table 1.3 **Differences between discrete and continuous distributions**

Discrete	Continuous
Variable limited to certain values	Variable not limited
Shape is usually stepped	Shape is usually smooth
Probabilities are equal to column heights	Probabilities are equal to areas under the curve
Sum of column heights = 1	Total area = 1

Example

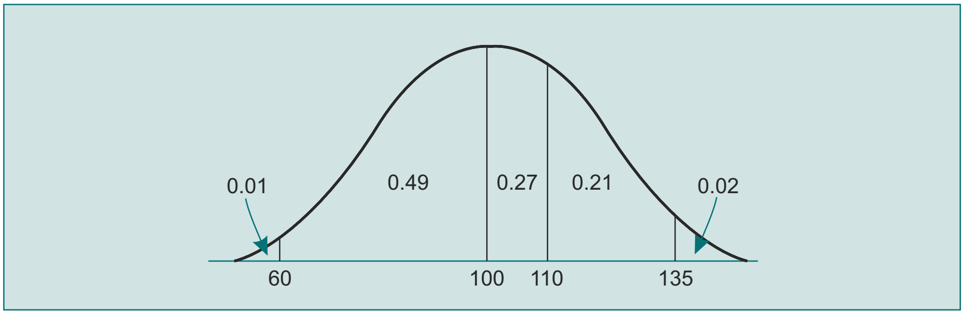

Figure 1.6 **The area under each part of the curve is shown. The total area is equal to 1.0**

Using the continuous distribution in Figure 1.6, what are the probabilities that a particular value of the variable falls within the following ranges?

1. $x \leq 60$
2. $x \leq 100$
3. $60 \leq x \leq 110$
4. $x \geq 135$
5. $x \geq 110$

Answers

1. $P(x \leq 60) = 0.01$
2. $P(x \leq 100) = 0.01 + 0.49 = 0.5$
3. $P(60 \leq x \leq 110) = 0.49 + 0.27 = 0.76$
4. $P(x \geq 135) = 0.02$
5. $P(x \geq 110) = 0.21 + 0.02 = 0.23$

In practice, the problems with the use of continuous distributions are, first, that one can never collect sufficient data, sufficiently accurately measured, to establish a continuous distribution. Second, were this possible, the accurate measurement of areas under the curve would be difficult. Their greatest practical use is where continuous distributions appear as *standard distributions*, a topic discussed in the next section.

1.5 Standard Distributions

The distribution of US sales data shown in Figure 1.2, Table 1.1, Table 1.2 and Figure 1.3 is an **observed distribution**. The data were collected, a histogram formed and that was the distribution. A **standard distribution** has a theoretical, rather than observational, base. It is a distribution that has been defined mathematically from a theoretical situation. The characteristics of the situation are expressed mathematically and the resulting situation constructed theoretically. When an actual situation resembling the theoretical one arises, the associated standard distribution is applied.

For example, one standard distribution, the **normal**, is derived from the following theoretical situation. A variable is generated by a process which should give the variable a constant value, but does not do so because it is subject to many small disturbances. As a result, the variable is distributed around the central value (*see* Figure 1.7). This situation (central value, many small disturbances) can be expressed mathematically and the resulting distribution can be anticipated mathematically (i.e. a formula describing the shape of the distribution can be found).

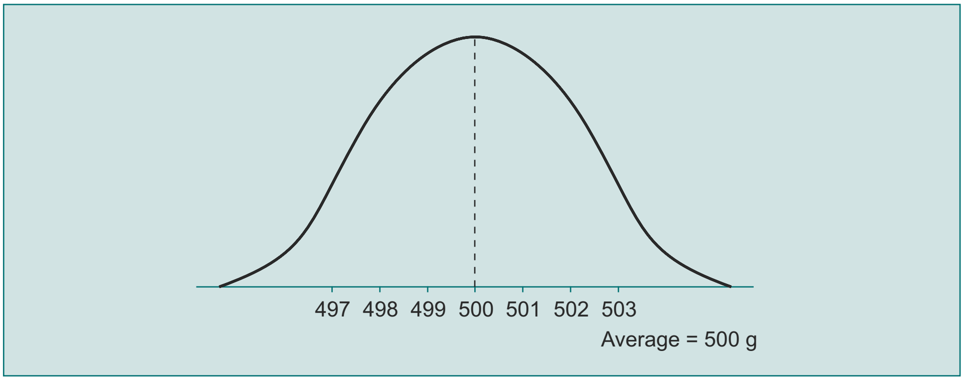

Figure 1.7 **Normal distribution of weights of loaves of bread**

If an actual situation appears to be like the theoretical, the normal distribution is applied. Analysis, similar to the probability calculations with the USA sales data, can then be carried out. Areas under parts of the curve can be found from the mathematical formula or, more easily, from the normal curve tables. The normal distribution would apply, for instance, to the lengths of machine-cut rods. The rods should all be of the same length, but are not because of the variation introduced by vibration, machine inaccuracies, the operator and other factors. A typical analysis might be to calculate the percentage of production likely to be outside engineering tolerances for the rods.

The normal distribution can be applied to many situations with similar characteristics. Other standard distributions relate to situations with different characteristics. Applying a standard distribution is an approximation. The actual situation is unlikely to match exactly the theoretical one on which the mathematics were based. However, this disadvantage is more than offset by the saving in data collection that the use of a standard distribution brings about. Observed distributions often entail a great deal of data collection. Not only must sufficient data be collected for the distribution to take shape, but also data must be collected individually for each and every situation.

In summary, using an observed distribution implies that data have been collected and histograms formed; using a standard distribution implies that the situation in which data are being generated resembles closely a theoretical situation for which a distribution has been constructed mathematically.

1.5.1 The Normal Distribution

The normal distribution, one of the most common, is now investigated in more detail. Figure 1.7 gives a rough idea of what it looks like in the case of weights of bread loaves. The principal features are that it is symmetrical and bell-shaped; it has just one hump (i.e. it is **unimodal**); the hump is at the average of the variable.

However, not all normal distributions are completely the same. Otherwise, they could not possibly represent both the weights of bread loaves (with an average value of 500 g and a spread of less than 10 g) and heights of male adults (with an average of 1.75 m and a spread of around 0.40 m). All normal curves share a number of common properties such as those mentioned above but they differ in that the populations they describe have different characteristics. Two factors, called **parameters**, capture these characteristics and are sufficient to distinguish one normal curve from another (and conversely specify exactly a normal curve). A parameter is defined as a measure describing some aspect of a population.

The first parameter is the **average** or mean of the distribution. Although the term 'average' has not been formally defined yet, it is no more than the expression in everyday use (e.g. the average of 2 and 4 is 3). Two normal distributions differing only by this parameter have precisely the same shape, but are located at different points along a horizontal scale.

The second parameter is the **standard deviation**. Its precise definition will be given later. It measures the dispersion, or spread, of the variable. In other words, some variables are clustered tightly about the average (such as the bread loaves). These distributions have a low standard deviation and their shape is narrow and high. Variables that are spread a long way from the average have a high standard deviation and their distribution is low and flat. Figure 1.8 shows examples of distributions with high and low standard deviations: salaries in a hospital have a large spread ranging from those for cleaners to those for consultants; salaries for teaching staff at a school have a much smaller spread.

A further characteristic of a normal distribution is related to the standard deviation (*see* Figure 1.9). The data refer to the weights of bread loaves with average weight 500 g and standard deviation 2 g.

The property of the normal distribution illustrated in Figure 1.9 is derived from the underlying mathematics, which are beyond the scope of this introduction. In any case, it is more important to be able to use the normal distribution than to prove its properties mathematically. The property applies whether the distribution is flat and wide or high and narrow, provided only that it is normal. Given such a property, it is possible to calculate the probabilities of events. The example below demonstrates how a standard distribution (in this case the normal) can be used in statistical analysis.

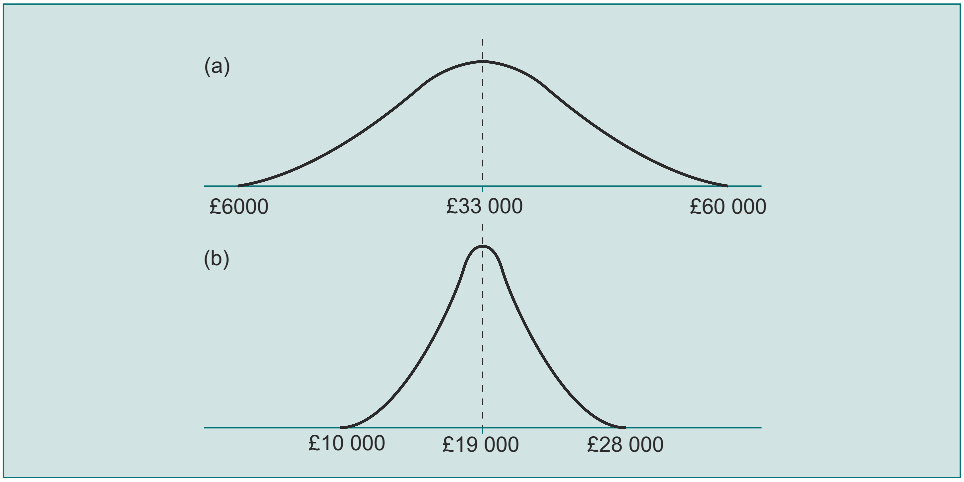

Figure 1.8 **Salaries: (a) hospital – high standard deviation; (b) school – low standard deviation**

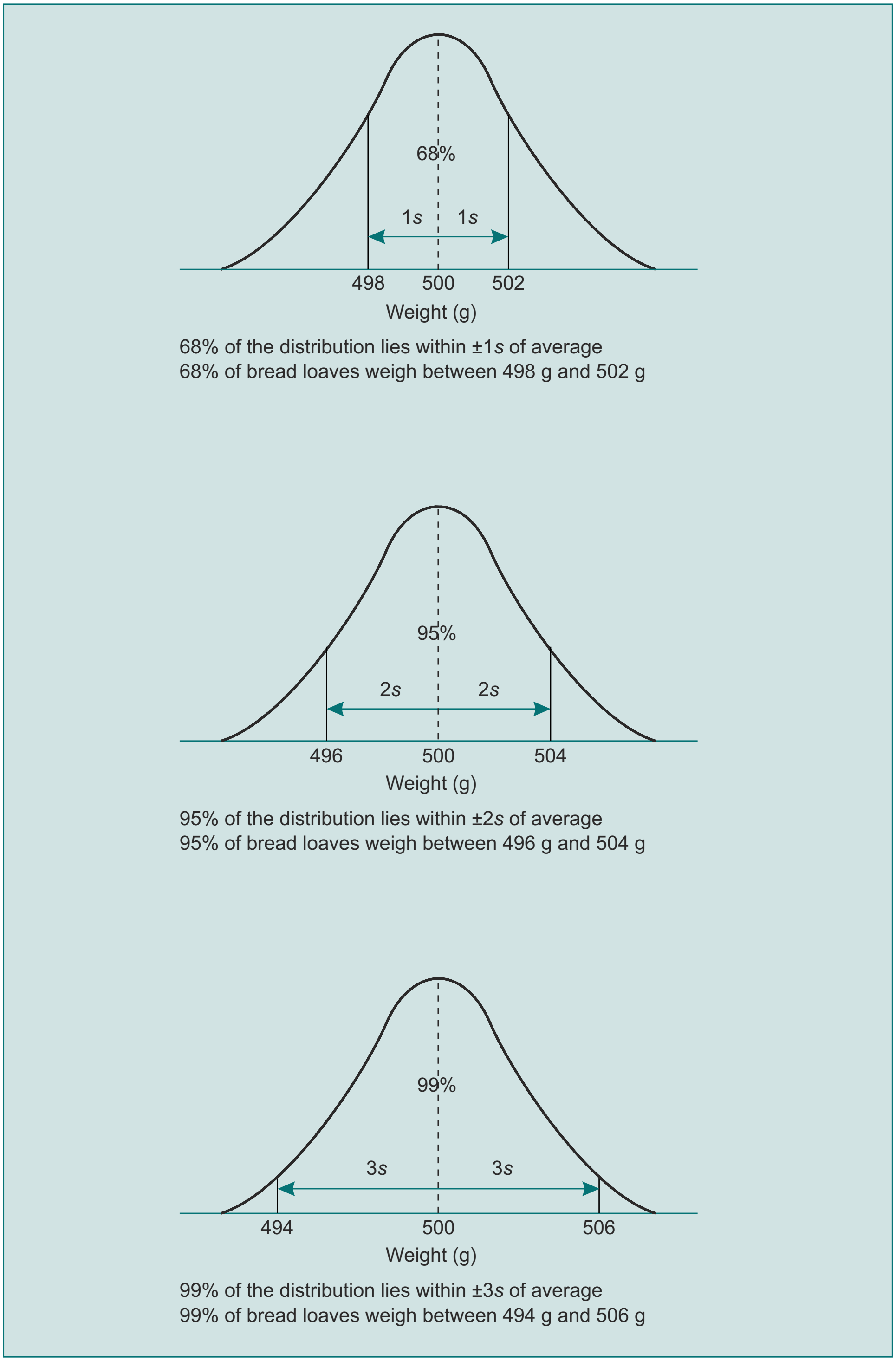

Figure 1.9 **Characteristics of the standard deviation (*s*) in a normal distribution**

Example

A machine is set to produce steel components of a given length. A sample of 1000 components is taken and their lengths measured. From the measurements the average and standard deviation of all components produced are estimated to be 2.96 cm and 0.025 cm respectively. Within what limits would 95 per cent of all components produced by the machine be expected to lie?

Take the following steps:

1. Assume that the lengths of all components produced follow a normal distribution. This is reasonable since this situation is typical of the circumstances in which normal distributions arise.
2. The parameters of the distribution are the average mean = 2.96 cm and the standard deviation = 0.025 cm. The distribution of the lengths of the components will therefore be as in Figure 1.10.

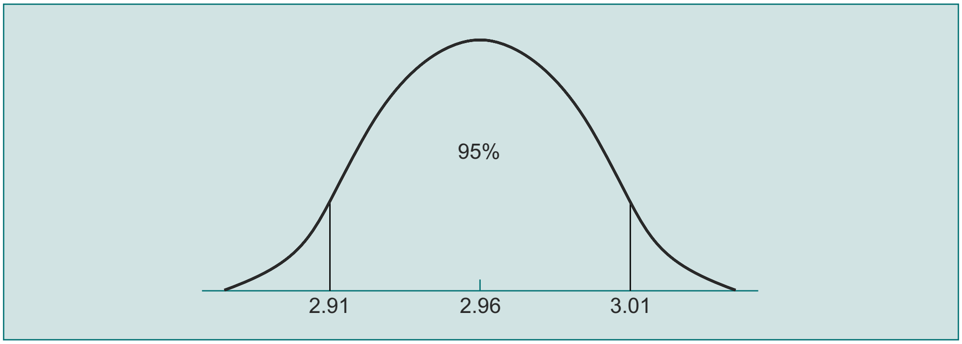

Figure 1.10 **Distribution of lengths of steel components**

There is a difference between the distribution of *all* components produced by the machine (the distribution of the **population**) and the distribution of the lengths of components in the sample. It is the former distribution which is of interest and which is shown in Figure 1.10. The sample has been used to estimate the parameters.

3. From the properties of the normal distribution stated above, 95 per cent of the distribution of the population (and therefore 95 per cent of all components produced) will be within two standard deviations of the average. Limits are $2.96 - (2 \times 0.025)$ and $2.96 + (2 \times 0.025)$, which give 2.91 cm and 3.01 cm.
   According to this estimate, 95 per cent of all production will lie between 2.91 cm and 3.01 cm.

1.6 Wrong Use of Statistics

Statistics are misused whenever statistical evidence is presented in such a way that it tends to lead to a false conclusion. The Advertising Standards Authority tries to protect the public from misleading advertising, but the manager has no similar protection against misleading management data. The presentation may mislead accidentally or deliberately. In the latter case, the misuse of statistics can be a creative art. Even so, it is possible to notice a few general types of misuse.

1.6.1 Definitions

Statistical expressions and the variables themselves may not have precise definitions. The user may assume the producer of the data is working with a different definition than is the case. By assuming a wrong definition, the user will draw a wrong conclusion. The statistical expression 'average' is capable of many interpretations. A firm of accountants advertises in its recruiting brochure that the average salary of qualified accountants in the firm is £44 200. A prospective employee may conclude that financially the firm is attractive to work for. A closer look shows that the accountants in the firm and their salaries are as follows:

3 partners	£86 000
8 senior accountants	£40 000
9 junior accountants	£34 000

The average salary could be:

$$\text{The mean} = \frac{(3 \times 86\,000) + (8 \times 40\,000) + (9 \times 34\,000)}{20} = £44\,200$$

The 'middle' value = £40 000

The most frequent value = £34 000

All the figures could legitimately be said to be the average salary. The firm has doubtless chosen the one that best suited its purposes. Even if it were certain that the correct statistical definition was being used, it would still be necessary to ask just how the variable (salary) is defined. Is share of profits included in the partners' salaries? Are bonuses included in the accountants' salaries? Are allowances (a car, for example) included in the accountants' salaries? If these items are removed, the situation might be:

3 partners	£50 000
8 senior accountants	£37 000
9 junior accountants	£32 400

The mean salary is now £36 880. Remuneration at this firm is suddenly not quite so attractive.

1.6.2 Graphics

Statistical pictures are intended to communicate data very rapidly. This speed means that first impressions are important. If the first impression is wrong then it is unlikely to be corrected.

There are many ways of representing data pictorially, but the most frequently used is probably the graph. If the scale of a graph is concealed or not shown at all, the wrong conclusion can be drawn. Figure 1.11 shows the sales figures for a company over the last three years. The company would appear to have been successful.

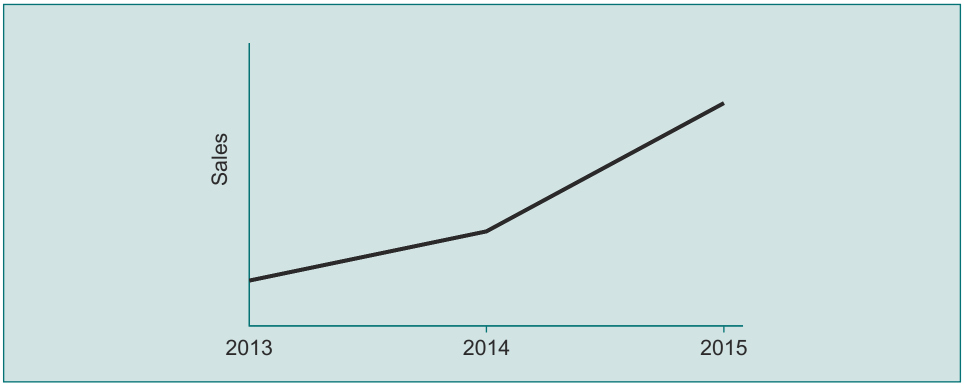

Figure 1.11 **Sales record (no scale)**

However, no scales are shown. In fact, the sales record has been:

2013	£11 250 000
2014	£11 400 000
2015	£11 650 000

A more informative graph showing the scale is given in Figure 1.12. Sales have hardly increased at all. Allowing for inflation, they have probably decreased in real terms.

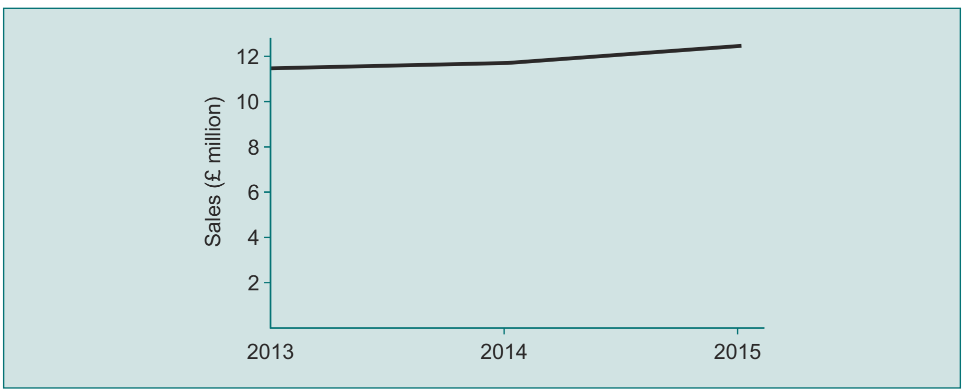

Figure 1.12 **Sales record (with scale)**

1.6.3 Sample Bias

Most statistical data are collected as a sample (i.e. they are just a small part of the total data available (the population)). Conclusions drawn from the sample are generalised to the population. The generalisation can be valid only to the extent that the sample is representative. If the sample is not representative then the wrong conclusions will be drawn. Sample bias can occur in three ways.

First, it arises in the collection of the data. The left-wing politician who states that 80 per cent of the letters he receives are against a policy of the right-wing government and concludes that a majority of all the electorate oppose the government on this issue is drawing a conclusion from a biased sample.

Second, sample bias arises through the questions that elicit the data. Questions such as 'Do you go to church regularly?' will provide unreliable information. There may be a tendency for people to exaggerate their attendance since, generally, it is regarded as a worthy thing to do. The word 'regularly' also causes problems. Twice a year, at Christmas and Easter, is regular. So is twice every Sunday. It would be difficult to draw any meaningful conclusions from the question as posed. The question should be more explicit in defining regularity.

Third, the sample information may be biased by the interviewer. For example, supermarket interviews about buying habits may be conducted by a young male interviewer who questions 50 shoppers. It would not be surprising if the resultant sample comprised a large proportion of young attractive females.

The techniques of sampling which can overcome most of these problems will be described later in the course.

1.6.4 Omissions

The statistics that are not given can be just as important as those that are. A television advertiser boasts that nine out of ten dogs prefer Bonzo dog food. The viewer may conclude that 90 per cent of all dogs prefer Bonzo to any other dog food. The conclusion might be different if it were known that:

(a) The sample size was exactly ten.
(b) The dogs had a choice of Bonzo or the cheapest dog food on the market.
(c) The sample quoted was the twelfth sample used and the first in which as many as nine dogs preferred Bonzo.

1.6.5 Logical Errors

Statistics allows conclusions about numbers to be drawn. Usually, however, it is the entities that lie behind the numbers that are of interest. Two of the most common ways for logical errors to be made are as follows.

First, the numbers may not be the same as the entities. For example, employee dissatisfaction is sometimes measured through staff turnover. It is the first that is being studied, but the numbers measure the second. The two may not always correspond. Financial analysts study the profit figures of companies in order to judge the profitability of the company. Profit figures are, however, just accounting measures and are no more than (hopefully, good) approximations to the 'true profitability' of the company, which is difficult both to define and to measure.

Second, conclusions about the numbers do not necessarily imply causal effects in the entities. For instance, there is a well-established relationship between the average salary of clergymen and the price of rum. The two variables move together and this can be verified statistically. However, this does not mean that clergymen support the price of rum or vice versa. The explanation is that the variables are related via a third factor, inflation. The variables have increased together as the cost of living has increased, but they are unlikely to be causally related. This consideration is important when decisions are based on statistical association. To take the example further, holding down clergymen's salaries in order to hold down the price of rum would work if the relationship were causal, but not if it were mere association.

1.6.6 Technical Errors

Mistakes occur where there is an insufficient understanding of even basic technicalities. An oft-quoted and simplistic case is that of a trade union leader stating his concern for the lower paid by saying that he would not rest until all his members earned more than the average salary for the union. (It may be that he was in fact making a very subtle statement.)

Another simple mistake is in the use of percentages. It would be wrong to suppose that, for example, a 20 per cent increase in productivity this year makes up for a 20 per cent decrease last year. If the index of productivity two years ago was 100, then a 20 per cent decrease makes it 80. The 20 per cent increase then makes it 96 (i.e. it has not been returned to its former level).

1.7 How to Spot Statistical Errors

Many types of statistical error can only be dealt with in the context of a particular quantitative technique, but there are several general questions which can help to uncover statistical errors and trickery. These questions should be posed whenever statistical evidence is used.

1.7.1 Who Is Providing the Evidence?

The law provides a good analogy. In a legal case the standing of a witness is an important consideration in evaluating evidence. One does not expect the defence counsel to volunteer information damaging to his/her client. In statistics also it is important to know who is providing the evidence. If the provider stands to gain from your acceptance of their conclusion, greater care is needed.

It is inconceivable that the makers of Bonzo dog food should ever declare 'We have long believed that Bonzo is the finest dog food available. However, recent tests with a random sample of 2000 dogs indicate that the product made by the Woof Corporation…'. On the other hand, a report on dog food by an independent consumer unit carries a greater likelihood of being reliable evidence.

1.7.2 Where Did the Data Come from?

In 2014 'on average British people took 2.38 baths per week, compared with 1.15 twenty years ago' reports a survey of people's washing habits carried out by a government department. On the surface this appears to be straightforward evidence, but how reliable is it?

Where did the data come from? One can assume not from personal observation. Most probably people were asked. Since not to bath frequently would be a shameful admission, answers may well be biased. The figure of 2.38 is likely to be higher than the true figure. Even so, a comparison with 20 years ago can still be made, but only provided the bias is the same now as then. It may not be. Where did the 20-year-old data come from? Most likely from a differently structured survey of different sample size, with different questions and in a different social environment. The comparison with 20 years ago, therefore, is also open to suspicion.

One is also misled in this case by the accuracy of the data. The figure of 2.38 suggests a high level of accuracy, completely unwarranted by the method of data collection. When numbers are presented to many decimal places, one should question the relevance of the claimed degree of accuracy.

1.7.3 Does It Pass the Common-Sense Test?

Experts in any subject sometimes can become so involved with their work that they see only the technicalities and not the main issues. Outsiders, inhibited by their lack of technical expertise, may suppress common-sense questions to the detriment of a project or piece of research. Anything that does not appear to make sense should be questioned.

An academic researcher investigated the relationship between total lifetime earnings and age at death, and found that the two variables were closely related. He concluded that poverty induces early death.

One may question the fact that he is basing a causal conclusion on a statistical association. Perhaps more importantly, an outsider may think that being alive longer gives more time to amass earnings and therefore it is at least as valid to conclude that the causality works in the opposite direction (i.e. an early death causes a person to have low total lifetime earnings). The researcher was so involved in his work and also probably had such a strong prior belief that poverty causes early death that he did not apply the common-sense test.

1.7.4 Has One of the Six Common Errors Been Committed?

Six of the more common types of statistical errors were described in the last section. Could one of them have been committed? Check through the six categories to see if one of them could apply:

(a) **Is there ambiguity of definition?** A statistical term (especially the average) capable of more than one interpretation may have been used.
(b) **Are the pictorial representations misleading?** Take a second look to see if other conclusions could be drawn. Especially check that scales have been included.
(c) **Is there sample bias?** When two samples are compared, is like being compared with like?
(d) **What is missing?** Is there any additional information which should have been included and which could change the conclusion?
(e) **Is there a logical error?** The numbers may not fully represent the entities they are intended to measure; a strong associative relationship may not be causal.

(f) **Is there a technical error?** Have statistical definitions/techniques/methods been properly applied? Answering this question will usually require a deeper theoretical knowledge of the subject.

Learning Summary

The purpose of this introduction has been twofold. The first aim has been to present some statistical concepts as a basis for more detailed study of the subject. All the concepts will be further explored. The second aim has been to encourage a healthy scepticism and atmosphere of constructive criticism, which are necessary when weighing statistical evidence.

The healthy scepticism can be brought to bear on applications of the concepts introduced so far as much as elsewhere in statistics. Probability and distributions can both be subject to misuse.

Logical errors are often made with **probability**. For example, suppose a questionnaire about marketing methods is sent to a selection of companies. From the 200 replies, it emerges that 48 of the respondents are not in the area of marketing. It also emerges that 30 are at junior levels within their companies. What is the probability that any particular questionnaire was filled in by someone neither in marketing nor at a senior level? It is tempting to suppose that:

$$\text{Probability} = \frac{48+30}{200} = 39\%$$

This is almost certainly wrong because of double counting. Some of the 48 non-marketers are also likely to be at a junior level. If 10 respondents were non-marketers *and* at a junior level, then:

$$\text{Probability} = \frac{48+30-10}{200} = 34\%$$

Only in the rare case where none of those at a junior level were outside the marketing area would the first calculation have been correct.

Figure 1.13 Civil servants' salaries

Graphical errors can frequently be seen with distributions. Figure 1.13 shows an observed distribution relating to the salaries of civil servants in a government department. The figures give a wrong impression of the spread of salaries because the class intervals are not all equal. One could be led to suppose that salaries are higher than they are. The lower bands are of width £8000 (0–8, 8–16, 16–24). The higher ones are of a much larger size. The distribution should be drawn with all the intervals of equal size, as in Figure 1.14.

Statistical concepts are open to misuse and wrong interpretation just as verbal reports are. The same vigilance should be exercised in the former as in the latter.

Figure 1.14 **Civil servants' salaries (amended)**

Review Questions

1.1 One of the reasons probability is important in statistics is that, if data being dealt with are in the form of a sample, any conclusions drawn cannot be 100 per cent certain. True or false?

1.2 A randomly selected card drawn from a pack of cards was an ace. It was not returned to the pack. What is the probability that a second card drawn will also be an ace?

A. 1/4
B. 1/13
C. 3/52
D. 1/17
E. 1/3

1.3 Which of the following statements are true?

A. The probability of an event is a number between 0 and 1.
B. Since nothing is ever certain, no event can have a probability equal to 1.
C. Classical statisticians take the view that subjective probability has no validity.
D. Bayesian statisticians take the view that only subjective probability has validity.

1.4 A coin is known to be unbiased (i.e. it is just as likely to come down 'heads' as 'tails'). It has just been tossed eight times and each time the result has been 'heads'. On the ninth throw, what is the probability that the result will be 'tails'?

A. Less than 1/2
B. 1/2
C. More than 1/2
D. 1

Questions 1.5–1.7 are based on the following information:

A train station's daily ticket sales (in £000) over the last quarter (= 13 weeks = 78 days) have been collected in histogram form as shown in Figure 1.15.

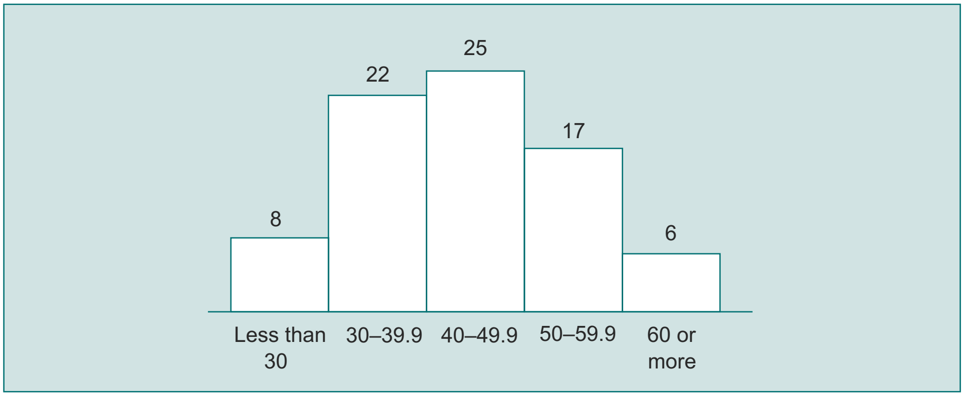

Figure 1.15 **Train ticket sales**

1.5 On how many days were sales not less than £50 000?

A. 17
B. 55
C. 23
D. 48

1.6 What is the probability that on any day sales are £60 000 or more?

A. 1/13
B. 23/78
C. 72/78
D. 0

1.7 What is the sales level that was exceeded on 90 per cent of all days?

A. £20 000
B. £30 000
C. £40 000
D. £50 000
E. £60 000

1.8 Which of the following statements about a normal distribution is true?

A. A normal distribution is another name for a standard distribution.
B. The normal distribution is an example of a standard distribution.
C. The normal distribution is a discrete distribution.
D. The normal distribution may or may not be symmetrical depending upon its parameters.

1.9 A normal distribution has mean 60 and standard deviation 10. What percentage of readings will be in the range 60–70?

A. 68%
B. 50%
C. 95%
D. 34%
E. 84%

1.10 A police checkpoint recorded the speeds of motorists over a one-week period. The speeds had a normal distribution with a mean 82 km/h and standard deviation 11 km/h. What speed was exceeded by 97.5 per cent of motorists?

A. 49
B. 60
C. 71
D. 104

Case Study 1.1: Airline Ticketing

As a first step towards planning new facilities at one of its city centre ticket offices, an airline has collected data on the length of time customers spend at a ticket desk (the service time). One hundred customers were investigated and the time in minutes each one was at an enquiry desk was measured. The data are shown below.

0.9	3.5	0.8	1.0	1.3	2.3	1.0	2.4	0.7	1.0
2.3	0.2	1.6	1.7	5.2	1.1	3.9	5.4	8.2	1.5
1.1	2.8	1.6	3.9	3.8	6.1	0.3	1.1	2.4	2.6
4.0	4.3	2.7	0.2	0.3	3.1	2.7	4.1	1.4	1.1
3.4	0.9	2.2	4.2	21.7	3.1	1.0	3.3	3.3	5.5
0.9	4.5	3.5	1.2	0.7	4.6	4.8	2.6	0.5	3.6
6.3	1.6	5.0	2.1	5.8	7.4	1.7	3.8	4.1	6.9
3.5	2.1	0.8	7.8	1.9	3.2	1.3	1.4	3.7	0.6
1.0	7.5	1.2	2.0	2.0	11.0	2.9	6.5	2.0	8.6
1.5	1.2	2.9	2.9	2.0	4.6	6.6	0.7	5.8	2.0

1 Classify the data in intervals one minute wide. Form a frequency histogram. What service time is likely to be exceeded by only 10 per cent of customers?

Case Study 1.2: JP Carruthers Co.

The JP Carruthers Co. is a medium-sized manufacturing firm. Its sales figures are about £220 million and its employment level has been around 1100 for the last 10 years. Most of its sales are in the car industry. JPC's profit last year was £14 480 000. It has always enjoyed a reputation for reliability and have generally been regarded as being well managed.

With few exceptions, JPC's direct labour force, numbering about 600, is represented by the TWU, the Transport Workers' Union. It is the practice in this industry to negotiate employee benefits on a company-wide basis, but to negotiate wages for each class of work in a plant separately. For years, however, this antiquated practice has been little more than a ritual. Supposedly, the system gives workers the opportunity to express their views, but the fact is that the wages settlement in the first group invariably sets the pattern for all other groups within a particular company. The Door Trim Line at JPC was the key group in last year's negotiations. Being first in line, the settlement in Door Trim would set the pattern for JPC that year.

Annie Smith is forewoman for the Door Trim Line. There are many variations of door trim and Annie's biggest job is to see that they get produced in the right mix. The work involved in making the trim is about the same regardless of the particular variety. That is to say, it is a straight piecework operation and the standard price is 72p per unit regardless of variety. The work itself, while mainly of an assembly nature, is quite intricate and requires a degree of skill.

Last year's negotiations started with the usual complaint from the union about piece prices in general. There was then, however, an unexpected move. Here is the union's demand for the Door Trim Line according to the minutes of the meeting:

> We'll come straight to the point. A price of 72p a unit is diabolical… A fair price is 80p.
> The women average about 71 units/day. Therefore, the 8p more that we want amounts to an average of £5.68 more per woman per day…
> This is the smallest increase we've demanded recently and we will not accept less than 80p.

(It was the long-standing practice in the plant to calculate output on an average daily basis. Although each person's output is in fact tallied daily, the bonus is paid on daily output averaged over the week. The idea is that this gives a person a better chance to recoup if she happens to have one or two bad days.)

The union's strategy in this meeting was a surprise. In the past the first demand was purposely out of line and neither side took it too seriously. This time their demand was in the same area as the kind of offer that JPC's management was contemplating.

At their first meeting following the session with the union, JPC's management heard the following points made by the accountant:

a. The union's figure of 71 units per day per person is correct. I checked it against the latest Production Report. It works out like this:

   Average weekly output for the year to date is 7100 units; thus, average daily output is 7100/5 =1420 units/day.

   The number of women directly employed on the line is 20, so that average daily output is 1420/20 = 71 units/day/woman.

b. The union's request amounts to an 11.1 per cent increase: (80 − 72)/72 × 100 = 11.1.

c. Direct labour at current rates is estimated at £26 million. Assuming an 11.1 per cent increase across the board, which, of course, is what we have to anticipate, total annual direct labour would increase by about £2.9 million: £26 000 000 × 11.1% = £2 886 000.

Prior to the negotiations management had thought that 7 per cent would be a reasonable offer, being approximately the rate at which productivity and inflation had been increasing in recent years. Privately they had set 10 per cent as the upper limit to their final offer. At this level they felt some scheme should be introduced as an incentive to better productivity, although they had not thought through the details of any such scheme.

As a result of the union's strategy, however, JPC's negotiating team decided not to hesitate any longer. Working late, they put together their 'best' package using the 10 per cent criterion. The main points of the plan were as follows:

a. Maintain the 72p per unit standard price but provide a bonus of 50p for each unit above a daily average of 61units/person.
b. Since the average output per day per person is 71, this implies that on average 10 bonus units per person per day would be paid.
c. The projected weekly cost then is £5612:
   $(71 \times 0.72) + (10 \times 0.50) = 56.12$
   $56.12 \times 5 \times 20 = £5612$
d. The current weekly cost then is £5112:
   $71 \times 0.72 \times 5 \times 20 = 5112$
e. This amounts to an average increase of £500 per week, slightly under the 10 per cent upper limit:
   $500/5112 \times 100 = 9.78\%$
f. The plan offers the additional advantage that the average worker gets 10 bonus units immediately, making the plan seem attractive.
g. Since the output does not vary much from week to week, and since the greatest improvement should come from those who are currently below average, the largest portion of any increase should come from units at the lower cost of 72p each. Those currently above average probably cannot improve very much. To the extent that this occurs, of course, there is a tendency to reduce the average cost below the 79p per unit that would result if no change at all occurs:
   $5612/(71 \times 5 \times 20) = 79.0\text{p}$

At this point management had to decide whether they should play all their cards at once or whether they should stick to the original plan of a 7 per cent offer. Two further issues had to be considered:

a. How good were the rates?
b. Could a productivity increase as suggested by the 9.8 per cent offer plan really be anticipated?

Annie Smith, the forewoman, was called into the meeting, and she gave the following information:

a. A few workers could improve their own average a little, but the rates were too tight for any significant movement in the daily outputs.
b. This didn't mean that everyone worked at the same level, but that individually they were all close to their own maximum capabilities.
c. A number did average fewer than 61 units per day. Of the few who could show a sustained improvement, most would be in this fewer-than-61 category.

This settled it. JPC decided to go into the meeting with their 'best' offer of 9.8 per cent. Next day the offer was made. The union asked for time to consider it and the next meeting was set for the following afternoon.

In the morning of the following day Annie Smith reported that her Production Performance Report (*see* Table 1.4) was missing. She did not know who had taken it but was pretty sure it was the union steward.

Table 1.4 **Production performance report**

Fiscal Week: 10	Cost Centre: 172	Foreperson: Smith
Employee Pay No.	**Av. Daily Output this Week**	**Av. Daily Output Y-T-D**
11	98	98
13	88	89
17	72	76
23	44	43
24	52	50
26	79	78
30	77	79
32	52	52
34	96	96
35	86	87
40	67	69
42	64	66
43	95	98
45	86	88
47	50	53
48	42	41
52	43	44
54	45	46
55	94	97
59	68	70
		Avg. 71
	AV. DAILY THIS WEEK – 1398	
	AV. DAILY YEAR-TO-DATE – 1420	

The next meeting with the union lasted only a few minutes. A union official stated his understanding of the offer and, after being assured that he had stated the details correctly, he announced that the union approved the plan and intended to recommend its acceptance to its membership. He also added that he expected this to serve as the basis for settlement in the other units as usual and that the whole wage negotiations could probably be completed in record time.

And that was that. Or was it? Some doubts remained in the minds of JPC's negotiating team. Why had the union been so quick to agree? Why had the Production Performance Report been stolen? While they were still puzzling over these questions, Annie Smith phoned to say that the Production Performance Report had been returned.

1 In the hope of satisfying their curiosity, the negotiating team asked Annie to bring the Report down to the office. Had any mistakes been made?

Was JPC's offer really 9.8 per cent? If not, what was the true offer?

Case Study 1.3: Newspaper Letters

The two attached letters appeared recently in a newspaper. In the first letter, Dr X concludes that dentists should not give anaesthetics. In the second, Mr Y concludes that dentists are the safest anaesthetists there are.

Danger in the Dental Chair

Sir– As a medically qualified anaesthetist responsible for a large number of dental anaesthetics I read (17 June) with great distress and despair of the death under an anaesthetic of Miss A.

It is a source of great concern to me that dentists are permitted to give anaesthetics. Any fool can give an intravenous injection, but considerable skill and experience is needed to handle an emergency occurring in anaesthetics.

For anyone, however qualified, however competent, to give an anaesthetic with no help whatsoever is an act of criminal folly; the BDA, BMA and all the medical defence societies would agree with this.

I call upon everyone to boycott anaesthetics given by a dentist under any circumstances.

Yours faithfully,

Dr X, Colchester, Essex.

A Dental Safety Record That Can't Be Matched

Sir– Dr X's feelings (Letters, 25 June) about the tragic death of Miss A will be shared by many, and they do him credit; but they have also led him astray.

Miss A was not anaesthetised; she was heavily sedated with a combination and dosage of drugs which produced a severe respiratory depression which the practitioner was unable to reverse.

In calling for a ban upon the giving of general anaesthetics by dentists, Dr X is on very unsafe ground. The possession of a medical degree does not of itself confer immunity from stupidity or negligence; many other people would still be alive if it did.

If Dr X consults the records produced by the Office of Population Censuses and Surveys, he will find that, overall, more deaths associated with dental anaesthesia occur when the anaesthetist is medically qualified than when he is a dentist.

Excluding the hospital service (where all anaesthetists are medically qualified but where nearly 50 per cent of deaths occur), medically qualified anaesthetists give 36 per cent of the dental anaesthetics; they have 45 per cent of the associated deaths. Not only a balance in favour of the dentist anaesthetist, but one which shows that mischance can occur to anyone, however skilled.

Not even Dr X, I think, would claim that all the deaths which occurred with medically qualified anaesthetists were due to misadventure, and all those which occurred with dentists were negligence.

However, these figures should be put in their proper perspective. In general dental practice and in the Community Dental Service, about 1.5 million anaesthetics are given each year. Over the last 15 years, deaths have averaged 4 a year. It is a safety record which cannot be matched by any other form of general anaesthesia.

Yours faithfully,

Mr Y, (President-Elect) Society for the Advancement of Anaesthesia in Dentistry.

1 Comment upon the evidence and reasoning (as given in the letters) that lead to these two conclusions.